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Transonic nozzle flows and free boundary problems for the full Euler equations

Gui-Qiang Chen [a], Jun Chen [b], Kyungwoo Song [c,*]

[a] *Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA*
[b] *Department of Mathematics, University of Wisconsin–Madison, Madison, WI 53706-1388, USA*
[c] *Department of Mathematics, Kyung Hee University, Hoeki-Dong, Dongdaemoon-Gu, Seoul 130-701, Republic of Korea*

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Abstract

We establish the existence and uniqueness of transonic flows with a transonic shock through a two-dimensional nozzle of slowly varying cross-sections. The transonic flow is governed by the steady, full Euler equations. Given an incoming smooth flow that is close to a constant supersonic state (i.e., smooth Cauchy data) at the entrance and the subsonic condition with nearly horizontal velocity at the exit of the nozzle, we prove that there exists a transonic flow whose downstream smooth subsonic region is separated by a smooth transonic shock from the upstream supersonic flow. This problem is approached by a one-phase free boundary problem in which the transonic shock is formulated as a free boundary. The full Euler equations are decomposed into an elliptic equation and a system of transport equations for the free boundary problem. An iteration scheme is developed and its fixed point is shown to exist, which is a solution of the free boundary problem, by combining some delicate estimates for the elliptic equation and the system of transport equations with the Schauder fixed point argument. The uniqueness of transonic nozzle flows is also established by employing the coordinate transformation of Euler–Lagrange type and detailed estimates of the solutions.

* Corresponding author.
E-mail addresses: gqchen@math.northwestern.edu (G.-Q. Chen), jchen@math.wisc.edu (J. Chen), kyusong@khu.ac.kr (K. Song).

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1. Introduction

We are concerned with transonic flow problems arising from a de Laval nozzle used for generating transonic gas flows (cf. [11,13]). In this paper, we study the existence and uniqueness of steady transonic flows with a transonic shock through a two-dimensional nozzle of slowly varying cross-sections for the full Euler equations. Given an incoming $C^{1,\alpha}$ flow that is close to a constant supersonic state (i.e., smooth Cauchy data) at the entrance and the subsonic condition with nearly horizontal velocity at the exit of the nozzle, we prove that there exists a transonic flow whose downstream smooth subsonic region is separated by a $C^{2,\alpha}$ transonic shock from the upstream $C^{1,\alpha}$ supersonic flow. This problem is approached by a one-phase free boundary problem in which the transonic shock is formulated as a free boundary for the Euler equations.

Such transonic problems have been studied under several different physical situations in the recent years. In [5–8], two nonlinear approaches have been developed to establish the existence and stability of transonic shocks in the $C^{1,\alpha}$ framework for the multidimensional Euler equations for steady potential fluids and applied to handling transonic flow problems in infinite channels and nozzles. Chen [9] also considered this problem in a channel for the two-dimensional steady Euler flows with a certain symmetry and obeying the Bernoulli law with a uniform Bernoulli constant (also see [10]). There are some related results for further simplified models: the unsteady transonic small disturbance equation in [1,3], the pressure-gradient system in [14,15], and the nonlinear wave system in [2] (also see the references cited therein).

For the full Euler equations, we first derive a second-order equation for the pressure and a system of transport equations for the velocity and the density. These equations are coupled together in their coefficients, and the equation for the pressure turns out to be elliptic in the subsonic region. Given an incoming $C^{1,\alpha}$ flow that is close to a uniform supersonic state at the entrance of the nozzle, $x = -1$, we first show that there exists a $C^{1,\alpha}$ supersonic flow up to $x = 1$ and then reformulate the transonic nozzle problem into a one-phase free boundary problem. That is, a part of the boundary of the subsonic region is a transonic shock which is regarded as a free boundary that is determined by an interaction between the flow of two sides of the shock. The conditions on the free boundary are the Rankine–Hugoniot relations between the supersonic and subsonic phases to ensure the flow is an entropy solution across the shock. To solve this free boundary problem, we have to determine both the free boundary and the subsonic phase defined in the region with the free boundary as a part of its boundary. We approach this problem by the iteration method via design of the linearized problems in fixed regions, making delicate estimates of the solutions, and performing a Schauder fixed point argument. In particular, we exploit the mixed boundary conditions to deal with the corner singularity caused by the nozzle boundary in the $C^{1,\alpha}$ framework to construct the subsonic flow in the iteration scheme. More precisely, given a fixed boundary $x = \psi(y)$ and a solution U, we solve a linear system of equations whose coefficients are from U to obtain a new solution $\widetilde{U}$ and extend it to the whole domain; then make delicate estimates of this solution and applying the Schauder fixed point argument to establish the existence of a fixed point U_*. For this fixed-point solution, we determine a fixed transonic shock $x = \phi_*(y)$ to the full Euler equations by the Rankine–Hugoniot relations between a perturbed supersonic state U_- and a subsonic state U_*. The uniqueness of transonic nozzle flows is also established by employing the coordinate transformation of Euler–Lagrange type and detailed estimates of the solutions when the incoming flow is $C^{2,\alpha}$.

One of the advantages in our analysis in this paper is in the context of the full Euler equations so that the solutions do not necessarily obey the Bernoulli law with a *uniform* Bernoulli constant (i.e., the Bernoulli constant is allowed to change for different fluid trajectories). Since we directly work on the nozzle domains with slowly varying cross-sections, we do not require additional symmetry of the solutions, unlike [6,9]. We remark that we assign the nearly horizontal velocity condition at the exit of the nozzle here, which is consistent with the subsonic condition at infinity in the infinite nozzle case for which the vertical velocity v should be zero for the uniformly nozzle boundary at infinity (cf. [8]); and the pressure condition at the exit of the nozzle is generally ill-posed.

This paper is organized as follows. In Section 2, we reformulate the transonic nozzle problem into a one-phase free boundary problem in the nozzle and state the main theorems. In Section 3, we derive linearized elliptic equations and linearized transport equations and make delicate estimates of these solutions. In Section 4, we develop the iteration scheme for solving the nonlinear problem and prove the existence of its fixed point by the Schauder fixed point argument and careful estimates of the iteration scheme. In Section 5, we establish the uniqueness of solutions for the transonic nozzle problem by using the coordinate transformation of Euler–Lagrange type and delicate estimates of the solutions. In Section 6, we use the method of characteristics to construct the supersonic solutions in the upstream region which is close to a uniform supersonic flow. This step enables us to reformulate the transonic nozzle problem into a one-phase free boundary problem in Section 2.

2. Mathematical setup and main theorems

Consider steady, adiabatic flows governed by the following full Euler equations:

$$\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u E + u p \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v E + v p \end{pmatrix}_y = 0, \tag{2.1}$$

where (u, v) is the velocity, ρ the density, p the pressure, and

$$E = \frac{1}{2}\left(u^2 + v^2\right) + \frac{p}{(\gamma - 1)\rho}$$

the energy with adiabatic exponent $\gamma > 1$. We denote the sonic speed by

$$c = \sqrt{\gamma p / \rho}.$$

Denote $U(x, y) = (u, v, p, \rho)(x, y)$. Then the full Euler equations can be written in the general form of conservation laws:

$$F(U)_x + G(U)_y = 0 \tag{2.2}$$

with $F(U) = (\rho u, \rho u^2 + p, \rho u v, \rho u E + u p)^\top$ and $G(U) = (\rho v, \rho u v, \rho v^2 + p, \rho v E + v p)^\top$. Furthermore, the full Euler equations can be also written into the following nondivergence form:

$$\rho u_x + u \rho_x + \rho v_y + v \rho_y = 0, \tag{2.3}$$

$$\rho u u_x + \rho v u_y + p_x = 0, \tag{2.4}$$

$$\rho u v_x + \rho v v_y + p_y = 0, \tag{2.5}$$

$$\gamma p u_x + u p_x + \gamma p v_y + v p_y = 0. \tag{2.6}$$

When a shock front $S := \{x = \psi(y)\}$ forms in the flow, the Rankine–Hugoniot relations are

$$\big[F(U)\big] - \psi'\big[G(U)\big] = 0 \tag{2.7}$$

for the states on both sides of the shock, where the bracket [ ] is the jump between the quantities of two states across the shock front. The Rankine–Hugoniot relations in (2.7) are equivalent to the following conditions:

$$\frac{d\psi}{dy} = \frac{[\rho u v]}{[\rho v^2 + p]}, \tag{2.8}$$

$$[\rho u] = \frac{[\rho u v][\rho v]}{[\rho v^2 + p]}, \tag{2.9}$$

$$\big[\rho u^2 + p\big] = \frac{[\rho u v]^2}{[\rho v^2 + p]}, \tag{2.10}$$

$$\left[\frac{\rho u}{2}\big(u^2 + v^2\big) + \frac{\gamma}{\gamma - 1} u p\right] = \left[\frac{\rho v}{2}\big(u^2 + v^2\big) + \frac{\gamma}{\gamma - 1} v p\right] \frac{[\rho u v]}{[\rho v^2 + p]}. \tag{2.11}$$

In our analysis below, we use (2.8) to locate the shock front and (2.9)–(2.11) to find the linearized boundary conditions on the shock front.

Let

$$U_\pm^0 = \big(u_\pm^0, v_\pm^0, p_\pm^0, \rho_\pm^0\big)$$

be two constant states, a supersonic state U_-^0 and a subsonic state U_+^0, respectively, separated by a steady shock front at $x = 0$. Then

$$U^0 = \begin{cases} U_-^0 = (u_-^0, v_-^0, p_-^0, \rho_-^0) & \text{if } x < 0, \\ U_+^0 = (u_+^0, v_+^0, p_+^0, \rho_+^0) & \text{if } x > 0 \end{cases}$$

is a transonic shock solution of (2.1) with (2.7) so that

$$\big(c_+^0\big)^2 - \big(u_+^0\big)^2 - \big(v_+^0\big)^2 > \delta_0, \qquad \big(c_-^0\big)^2 - \big(u_-^0\big)^2 - \big(v_-^0\big)^2 < -\delta_0 \tag{2.12}$$

for some $\delta_0 > 0$, where $c_0^\pm = \sqrt{\gamma p_\pm^0 / \rho_\pm^0}$.

In particular, when $v_-^0 = v_+^0 = 0$, the Rankine–Hugoniot relations in (2.7) imply

$$\psi' = 0,$$

$$\big[\rho^0 u^0\big] = 0, \tag{2.13}$$

$$\big[\rho^0 \big(u^0\big)^2 + p^0\big] = 0, \tag{2.14}$$

$$\left[\rho^0 u^0\left(\frac{\gamma p^0}{(\gamma-1)\rho^0}+\frac{(u^0)^2}{2}\right)\right]=0. \tag{2.15}$$

Conditions (2.12)–(2.15) imply

$$u_+^0 < u_-^0, \qquad p_+^0 > p_-^0, \qquad \rho_+^0 > \rho_-^0. \tag{2.16}$$

This is the entropy condition for the constant solution.

We are interested in the existence and uniqueness of transonic flows through a two-dimensional nozzle of slowly varying cross-sections. Let $\varepsilon > 0$ be sufficiently small and $N > 1$. For concreteness, we define the domains by

$$\Omega = \big\{(x, y)\colon\ -1 < x < N,\ \zeta_2(x) < y < \zeta_3(x)\big\}, \tag{2.17}$$

$$\Omega_1 = \Omega \cap \{-1 < x < 1\}, \tag{2.18}$$

where

$$\begin{cases} \zeta_2(x) = 0 & \text{on } [-1, 1] \cup [N-1, N], \\ \zeta_3(x) = b & \text{on } [-1, 1] \cup [N-1, N], \end{cases}$$

with

$$\big\|\zeta_2(x)\big\|_{C^{2,\alpha}([-1,N])} + \big\|\zeta_3(x) - b\big\|_{C^{2,\alpha}([-1,N])} < \varepsilon.$$

Since our solutions will be near the background solution U^0, they will automatically satisfy the entropy condition since (2.16). In particular, when U satisfies

$$\big\|U - U_+^0\big\|_{C^{1,\alpha}(\Omega_+)} < C\varepsilon$$

for some constant $C > 0$, then U stays subsonic in the subsonic region Ω_+ in Ω.

Now we set up the transonic nozzle problem. Let ν be the outer unit normal to $\partial\Omega$.

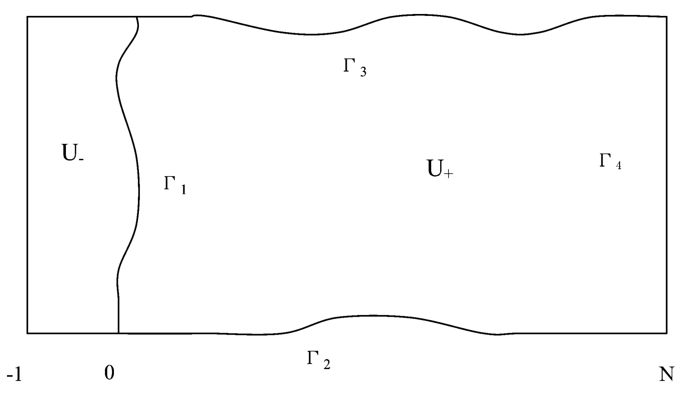

Fig. 1. Domain for the nozzle problem.

Problem 2.1 *(Transonic nozzle problem).* Given a uniform supersonic flow U_0^- and a smooth incoming flow close to U_0^- at the entrance, find a transonic flow U, which is supersonic after passing the entrance and subsonic with nearly horizontal velocity at the exit of the nozzle, separated by a transonic shock $x = \psi(y)$ with $\psi(0) = 0$ for the following problem of initial-boundary value type in an impermeable nozzle:

$$U|_{x=-1} = \Phi_0(y) := (u_0, v_0, p_0, \rho_0)(y) \in C^{1,\alpha}(0, b), \tag{2.19}$$

$$(u, v) \cdot \nu|_{\{y=\zeta_2(x),\zeta_3(x)\}} = 0, \tag{2.20}$$

$$v|_{x=N} = h, \qquad \left(u^2 + h^2 - c^2\right)\big|_{x=N} < 0, \tag{2.21}$$

with

$$\left\|\Phi_0 - U_-^0\right\|_{C^{1,\alpha}(0,b)} + \|h\|_{C^{1,\alpha}(0,b)} < \varepsilon, \qquad (u_0, p_0, \rho_0)_y\big|_{y=0,b} = 0, \qquad h\big|_{y=0,b} = 0 \tag{2.22}$$

for some small $\varepsilon > 0$.

The conditions in (2.22) are technical to ensure the compatibility of solutions in $C^{1,\alpha}$ at the points and the nearly horizontal velocity at the exit. The condition on Γ_4 is consistent with the condition at the infinite downstream for the infinite nozzle problem (see [8]). In order to solve this nozzle problem, we first establish the existence and uniqueness of supersonic flows in the upstream region Ω_1 via the method of characteristics. The following theorem is established in Section 6.

Theorem 2.1. *Given small $\varepsilon > 0$, there exist a constant $C_0 > 0$ and a unique supersonic solution $U_- = (u_-, v_-, p_-, \rho_-)(x, y) \in C^{1,\alpha}(\Omega_1)$ of Problem* (2.3)–(2.6) *and* (2.19)–(2.22) *such that*

$$\left\|U_- - U_-^0\right\|_{C^{1,\alpha}(\Omega_1)} < C_0\varepsilon, \tag{2.23}$$

$$(u_-, p_-, \rho_-)_y\big|_{\{y=0,b\}\cap\Omega_1} = 0. \tag{2.24}$$

In addition, if the Cauchy data and the nozzle satisfy

$$\left\|\Phi_0 - U_-^0\right\|_{C^{2,\alpha}(0,b)} + \left\|(\zeta_2, \zeta_3 - b)\right\|_{C^{3,\alpha}(-1,N)} < \varepsilon, \tag{2.25}$$

then the solution $U_-(x, y) \in C^{2,\alpha}$ also satisfies

$$\left\|U_- - U_-^0\right\|_{C^{2,\alpha}(\Omega_1)} < C_0\varepsilon. \tag{2.26}$$

With Theorem 2.1, we can reformulate Problem 2.1 into the following one-phase free boundary problem.

Problem 2.2. Given a supersonic solution $U_-(x, y) = (u_-, v_-, p_-, \rho_-)(x, y)$ of (2.3)–(2.6), (2.19), and (2.20) satisfying (2.23) and (2.24) for some small constant $\varepsilon > 0$, find a subsonic flow $U(x, y)$ in the downstream separated by a transonic shock front $x = \psi(y)$ with $\psi(0) = 0$ satisfying (2.20)–(2.22).

To solve this free boundary problem, we develop an iteration scheme to solve first a fixed boundary problem to determine the downstream flow and solve then the ordinary differential equation (2.8) to determine the location of the shock front. For the uniqueness of the shock, it is required that the shock pass through a specific point, say the origin $O=(0,0)$, in Problem 2.2. Then combining some delicate estimates of the iteration scheme with the Schauder fixed point argument yields the solution of a free boundary problem.

To determine a fixed boundary problem in the iteration scheme, we determine our domain first. The domain Ω is a nozzle that is a perturbation of a two-dimensional channel, and the elliptic region

$$\Omega_+ = \{(x,y)\colon\ \psi(y) < x < N,\ \zeta_2(x) < y < \zeta_3(x)\} \subset \Omega$$

is bounded by

$$\Gamma_1\colon\quad x=\psi(y),\qquad \Gamma_2\colon\quad y=\zeta_2(x),\qquad \Gamma_3\colon\quad y=\zeta_3(x),\qquad \Gamma_4\colon\quad x=N.$$

We assign the boundary conditions as follows: the Rankine–Hugoniot relations (2.7) on Γ_1; the slip condition

$$(u,v)\cdot\nu = 0 \tag{2.27}$$

on $\Gamma_2\cup\Gamma_3$; and $v=h$ on Γ_4.

For convenience, three intervals are defined by

$$I_1=\{y=0,\ -1\leqslant x\leqslant 1\},\qquad I_2=\{y=b,\ -1\leqslant x\leqslant 1\},\qquad J=\{y=b,\ |x|\leqslant 1/4\}.$$

Denote

$$\begin{aligned}\mathcal{M} = \Big\{&U_-\in C^{1,\alpha}(\Omega_1)\colon\ \big\|U_- - U_-^0\big\|_{C^{1,\alpha}(\Omega)} < C_0\varepsilon,\ (u_-,p_-,\rho_-)_y\big|_{O\cup J} = 0,\\ &(u_-,v_-)\cdot\nu = 0 \text{ on } I_1\cup I_2\Big\}.\end{aligned}$$

Here the condition $(u_-,p_-,\rho_-)_y = 0$ on $O\cup J$ is ensured by Theorem 2.1 and will be used for the reflection around the corners. Then we have

Theorem 2.2. *Let $\varepsilon>0$ be small and $U_-\in\mathcal{M}$. Then*

(i) *Problem* 2.2 *has a solution U and a shock front ψ such that the Rankine–Hugoniot relations* (2.7) *hold and*

$$\big\|U-U_+^0\big\|_{C^{1,\alpha}(\Omega)} < C\varepsilon, \tag{2.28}$$

$$\|\psi\|_{C^{2,\alpha}([0,b])} < C\varepsilon \tag{2.29}$$

for some constant C independent of ε;

(ii) *The solution for Problem* 2.2 *is unique provided additionally that conditions* (2.25) *and $\|h\|_{C^{2,\alpha}(0,b)} < \varepsilon$ hold.*

In Sections 3–5, we solve Problem 2.2 to establish Theorem 2.2 in detail, and we then prove Theorem 2.1 in Section 6. Then the nozzle problem is solved by combining Theorem 2.2 with Theorem 2.1.

3. Linearized equations with linearized boundary conditions

In this section, we first develop a linearization procedure for the nonlinear problem (2.3)–(2.7), (2.21), and (2.27), and we then establish the existence of solutions of the linearized problem to approach Problem 2.2.

For $\delta \in (0, \delta_0)$, define the δ-neighborhood Σ_δ of U_+^0 by

$$\Sigma_\delta = \left\{U \in C^{1,\alpha}(\Omega)\colon\ \left\|U - U_+^0\right\|_{C^{1,\alpha}(\Omega)} < \delta,\ v = \zeta_j' u \text{ on } \Gamma_j \text{ for } j = 2, 3, \text{ and } v = h \text{ on } \Gamma_4\right\}.$$

We now explain the iteration scheme to obtain the solutions δU from the linearized elliptic equation and the linearized transport equations with linearized boundary conditions.

Define

$$\mathcal{F} = \left\{\phi \in C^{2,\alpha}(0, b)\colon\ \phi(0) = \phi'(0) = 0,\ \|\phi\|_{C^{2,\alpha}(0,b)} \leqslant \eta_0\right\}$$

for some small $\eta_0 > 0$.

For any $U \in \Sigma_\delta$ and $\phi \in \mathcal{F}$, we solve the linearized problem on the fixed region with a fixed boundary part formed by ϕ: we first identify an equation for δp and prove the existence and uniqueness of the solution by the standard elliptic theory, and we then investigate δu, δv, and $\delta\rho$ in a linear system of transport equations.

First, we perform $\partial_x(2.4) + \partial_y(2.5) - \frac{\rho}{\gamma p}(u\partial_x + v\partial_y)(2.6)$ to obtain the following second-order linear equation:

$$\begin{aligned}
&\left(c^2 - u^2\right)(\delta p)_{xx} - 2uv(\delta p)_{xy} + \left(c^2 - v^2\right)(\delta p)_{yy} \\
&\quad - \big((\gamma+1)uu_x + vu_y + \gamma uv_y\big)(\delta p)_x - \big(\gamma vu_x + uv_x + (\gamma+1)vv_y\big)(\delta p)_y \\
&\quad + c^2\big((\rho u)_x u_x + (\rho u)_y v_x + (\rho v)_x u_y + (\rho v)_y v_y\big) = 0,
\end{aligned} \tag{3.1}$$

which is elliptic in the subsonic region.

To solve the elliptic problem, we impose the following proper boundary conditions:

on Γ_1, $\delta p = g_2$, where g_2 will be specified later;

on the boundaries Γ_j, $j = 2, 3$, we have the Neumann condition:

$$(\delta p)_\nu = (-1)^j \rho u^2 \zeta_j'' / \sqrt{1 + (\zeta_j')^2} \quad \text{on } \Gamma_j,\ j = 2, 3,$$

by (2.27), (2.4), and (2.5).

Notice that, during the process of obtaining the elliptic equation (3.1), we differentiate (2.6) by $u\partial_x + v\partial_y$. Therefore, (3.1) is not equivalent to (2.6) unless we impose equality (2.6) on Γ_4. This leads to a boundary condition for δp on Γ_4:

$$(\delta p)_x = \frac{c^2 \rho u}{c^2 - u^2}\left(-\frac{v}{u} u_y + v_y + \frac{v}{c^2 \rho} p_y\right).$$

Second, the system of linear transport equations is

$$\rho u(\delta u)_x + \rho v(\delta u)_y + (\delta p)_x = 0, \tag{3.2}$$

$$\rho u(\delta v)_x + \rho v(\delta v)_y + (\delta p)_y - K(x, y)\rho u\bigl(\delta u - u + u_+^0\bigr) = 0, \tag{3.3}$$

$$u(\delta\rho)_x + v(\delta\rho)_y - \frac{u}{c^2}(\delta p)_x - \frac{v}{c^2}(\delta p)_y = 0, \tag{3.4}$$

where δp is obtained in Eq. (3.1),

$$K(x, y) = \zeta_2''(x)\frac{\zeta_3(x) - y}{\zeta_3(x) - \zeta_2(x)} + \zeta_3''(x)\frac{y - \zeta_2(x)}{\zeta_3(x) - \zeta_2(x)},$$

and Eq. (3.4) is obtained from (2.3) $- \frac{\rho}{\gamma p}$(2.6).

We now explain the extra term containing a new function $K(x, y)$ in (3.3) which is not shown in the original full Euler equations. We know that

$$v/u = \zeta_j' \quad \text{on } \Gamma_j,\ j = 2, 3, \tag{3.5}$$

in the original Euler equations (2.3)–(2.6). For U satisfying (3.5), we require the next step linear solution δU to satisfy

$$\frac{\delta v}{\delta u + u_+^0} = \zeta_j' \quad \text{on } \Gamma_j,\ j = 2, 3.$$

This turns out to be true if we consider (3.2) and (3.3) along the fluid trajectory on $\Gamma_2 \cup \Gamma_3$.

To solve Eqs. (3.1)–(3.4) in a subsonic region, we need boundary conditions of δU in the region. We first define approximate Rankine–Hugoniot relations of (2.9)–(2.11).

Denote the tangential derivative along the shock front:

$$D_\tau := \frac{[\rho uv]}{[\rho v^2 + p]}\partial_x + \partial_y. \tag{3.6}$$

We now identify an approximate boundary condition of the Rankine–Hugoniot relations by taking the derivative of (2.9)–(2.11) for U with respect to the tangential direction τ of the shock front. Then the approximate boundary conditions are

$$\rho(\delta u)_\tau + u(\delta\rho)_\tau = (\rho_- u_-)_\tau + \left(\frac{[\rho v][\rho uv]}{[\rho v^2 + p]}\right)_\tau := (\rho_- u_-)_\tau + q_1,$$

$$\begin{aligned} 2\rho u(\delta u)_\tau + (\delta p)_\tau + u^2(\delta\rho)_\tau &= \bigl(\rho_-(u_-)^2 + p_-\bigr)_\tau + \left(\frac{[\rho uv]^2}{[\rho v^2 + p]}\right)_\tau \\ &:= \bigl(\rho_-(u_-)^2 + p_-\bigr)_\tau + q_2, \end{aligned}$$

$$\begin{aligned} &\left(\frac{3}{2}\rho u^2 + \frac{\gamma}{\gamma - 1}p\right)(\delta u)_\tau + \frac{\gamma}{\gamma - 1}u(\delta p)_\tau + \frac{u^3}{2}(\delta\rho)_\tau \\ &\quad = \left(\frac{1}{2}\rho_- u_-\bigl(u_-^2 + v_-^2\bigr) + \frac{\gamma}{\gamma - 1}u_- p_- - \frac{1}{2}\rho u v^2\right)_\tau + \left(\frac{[\frac{1}{2}\rho v(u^2 + v^2) + \frac{\gamma}{\gamma-1}vp][\rho uv]}{[\rho v^2 + p]}\right)_\tau \\ &\quad := \left(\frac{1}{2}\rho_-(u_-)^3 + \frac{\gamma}{\gamma - 1}u_- p_-\right)_\tau + q_3, \end{aligned}$$

where each q_i satisfies

$$\|q_i\|_{C^\alpha(\Omega)} \leqslant C\big(\varepsilon + \big\|U - U_+^0\big\|^2_{C^{1,\alpha}(\Omega)}\big) < C\big(\varepsilon + \delta^2\big),$$

and $q_i = 0$ on $I_1 \cup I_2$ for $i = 1, 2, 3$. This system of approximate conditions is equivalent to

$$BW_\tau = G_\tau + Q,$$

where

$$B = \begin{pmatrix} \rho & 0 & u \\ 2\rho u & 1 & u^2 \\ \frac{3\rho u^2}{2} + \frac{\gamma}{\gamma-1}p & \frac{\gamma}{\gamma-1}u & \frac{u^3}{2} \end{pmatrix}, \qquad W = \begin{pmatrix} \delta u \\ \delta p \\ \delta\rho \end{pmatrix},$$

$$G = \begin{pmatrix} \rho_- u_- \\ \rho_-(u_-)^2 + p_- \\ \frac{1}{2}\rho_-(u_-)^3 + \frac{\gamma}{\gamma-1}u_- p_- \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

Since $\det B \neq 0$ in a subsonic region, W_τ expressed by U and U^- satisfies

$$\|W_\tau\|_{C^\alpha(\Gamma_1)} \leqslant C\big(\varepsilon + \|Q\|_{C^\alpha(\Omega)}\big).$$

Integrating W_τ along the boundary with the starting point at the origin, we have the boundary conditions of

$$\delta u = g_1, \qquad \delta p = g_2, \qquad \delta\rho = g_3,$$

where g_i, $i = 1, 2, 3$, are independent of δU and satisfy

$$\|g_i\|_{C^{1,\alpha}(\Gamma_1)} \leqslant C\big(\varepsilon + \big\|U - U_+^0\big\|^2_{C^{1,\alpha}(\Omega)}\big) \leqslant C\big(\varepsilon + \delta^2\big) \tag{3.7}$$

for some constant $C > 0$.

So far, we have identified the boundary conditions on Γ_1 of W except for the normal component of the velocity, since it is degenerate in the linearization of the Rankine–Hugoniot relations. Instead of the boundary condition of δv on Γ_1, we impose the condition v on Γ_4:

$$v = h \quad \text{on } \Gamma_4$$

such that $\|h\|_{C^{1,\alpha}(0,b)} < \varepsilon$. Then, from Eq. (3.3), we can solve δv along the characteristics of the operator $D_{\bar{s}} := \rho(-u)\partial_x + \rho(-v)\partial_y$.

Under these proper boundary conditions on the subsonic region, we first obtain the existence result for the linear second-order elliptic equation.

Lemma 3.1. *Let $U_- \in \mathcal{M}$, $U \in \Sigma_\delta$, and $\phi \in \mathcal{F}$. Then there exists a unique solution $\delta p \in C^{1,\alpha}(\overline{\Omega_+})$ to the second-order linear elliptic equation* (3.1) *with the mixed boundary conditions*:

$$\delta p = g_2 \quad on\ \Gamma_1,$$
$$(\delta p)_\nu = \rho u^2 \zeta_2'' / \sqrt{1 + (\zeta_2')^2} \quad on\ \Gamma_2,$$
$$(\delta p)_\nu = -\rho u^2 \zeta_3'' / \sqrt{1 + (\zeta_3')^2} \quad on\ \Gamma_3,$$
$$(\delta p)_\nu = (\delta p)_x = \frac{c^2 \rho u}{c^2 - u^2} \left(-\frac{v}{u} u_y + v_y + \frac{v}{c^2 \rho} p_y \right) \quad on\ \Gamma_4.$$

Furthermore, for some $C > 0$,

$$\|\delta p\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\big(\varepsilon + \|g_2\|_{C^{1,\alpha}(\Gamma_1)} + \varepsilon \big\|U - U_+^0\big\|_{C^{1,\alpha}(\Omega_+)} + \big\|U - U_+^0\big\|^2_{C^{1,\alpha}(\Omega_+)}\big). \tag{3.8}$$

Proof. For simplicity, we use the standard index notations to rewrite the elliptic equation. Then the elliptic equation (3.1) is

$$a_{ij}(\delta p)_{x_i x_j} + b_i(\delta p)_{x_i} + r = \big(a_{ij}(\delta p)_{x_i}\big)_{x_j} + \big(b_i - (a_{ij})_{x_j}\big)(\delta p)_{x_i} + r,$$

where $a_{ij} \in C^{1,\alpha}(\Omega_+)$ and $b_i - (a_{ij})_{x_j} \in C^\alpha(\Omega_+)$ for $i, j = 1, 2$, and $r \in C^\alpha(\Omega_+)$ with

$$\|r\|_{C^\alpha(\Omega_+)} \leqslant C\big\|U - U_+^0\big\|^2_{C^{1,\alpha}(\Omega_+)}.$$

Then, by the standard theory of elliptic equations of divergence form, there exists a unique solution δp with

$$\delta p \in C^{1,\alpha}\big(\overline{\Omega_+} \setminus \{O, P_1, P_2, P_3\}\big),$$

where P_1 is the intersection point of Γ_1 and Γ_3, P_2 of Γ_2 and Γ_4, and P_3 of Γ_3 and Γ_4.

In order to obtain the $C^{1,\alpha}$ regularity up to the boundary, we perform the local reflection with respect to $y = 0$ and $y = b$. After a local reflection at the two points O and P_j with respect to $y = 0$ and $y = b$, we have

$$a_{ij} \in C^\alpha(\Omega_0), \qquad b_i - (a_{ij})_{x_j} \in L^\infty(\Omega_0), \qquad r \in L^\infty(\Omega_0),$$

where Ω_0 is a locally extended region of Ω_+ around the points O and P_j, $j = 1, 2, 3$, along the line segment $y = 0$ and b. With this regularity for the coefficients, we can conclude

$$\delta p \in C^{1,\alpha}(\overline{\Omega_+})$$

if the compatibility conditions of δp at the origin O and P_j, $j = 1, 2, 3$, can be verified by the local reflection. It is clear that δp is $C^{1,\alpha}$ around the two points P_2 and P_3 by the local reflection.

We note that the tangential derivative in (3.6) along the shock front becomes $D_\tau = \partial_y$ on $I_1 \cup I_2$. For $U_- \in \mathcal{M}$ and $\phi \in \mathcal{F}$, we find $(\delta U)_\tau = 0$ since $v = 0$ and $v_- = 0$ on $I_1 \cup I_2$, which implies $(\delta U)_y = 0$ there.

Around the points O and P_1 on $\Gamma_2 \cup \Gamma_3$, the normal derivative of δp becomes zero. Thus we have the compatibility conditions which enable us to do the local reflection over $\Gamma_2 \cup \Gamma_3$ around O and P_1, respectively. $\square$

We now establish the existence of solutions for (u, v, ρ) of the transport equations.

Lemma 3.2. *Given $U \in \Sigma_\delta$, there exists a solution $(\delta u, \delta v, \delta\rho) \in C^{1,\alpha}(\Omega_+)$ of the linear transport equations* (3.2)–(3.4) *satisfying*

$$\big\|(\delta u, \delta v, \delta\rho)\big\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\left(\varepsilon + \sum_{i=1}^{3} \|g_i\|_{C^{1,\alpha}(\Gamma_1)} + \varepsilon\|\delta U\|_{C^{1,\alpha}(\Omega_+)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\right) \tag{3.9}$$

for some constant $C > 0$.

Proof. We note that the linear elliptic problem (3.1) for δp is

$$\big(c^2 - u^2\big)(\delta p)_{xx} - 2uv(\delta p)_{xy} + \big(c^2 - v^2\big)(\delta p)_{yy} = f_0\big(u, v, p, \rho, Du, Dv, D\rho, D(\delta p)\big) := A$$

with $f_0 \in C^\alpha(\Omega_+)$. Then $\delta p \in C^{2,\alpha}(\Omega_+)$. Define

$$D_s = \rho u \partial_x + \rho v \partial_y.$$

From (3.2) and (3.3),

$$D_s(\delta u) + (\delta p)_x = 0,$$
$$D_{\bar{s}}(\delta v) - (\delta p)_y + K\rho u\big(\delta u - u + u_+^0\big) = 0.$$

Integration along the characteristics of D_s and $D_{\bar{s}}$ yields

$$\|\delta u\|_{C^\alpha(\Omega_+)} \leqslant C\big(\|g_1\|_{C^{1,\alpha}(\Gamma_1)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\big),$$
$$\|\delta v\|_{C^\alpha(\Omega_+)} \leqslant C\big(\varepsilon + \varepsilon\|\delta u\|_{C^\alpha(\Omega_+)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\big)$$

for $\delta p \in C^{1,\alpha}(\Omega_+)$ in Lemma 3.1.

We now consider the higher regularity of δu and δv. Taking the derivative on both sides of (3.2) with respect to y, we find

$$D_s(\delta u)_y + (\delta p)_{xy} + E_1(\delta u)_y + E_2(\delta p)_x = 0 \tag{3.10}$$

for some functions $E_1, E_2 \in C^\alpha(\Omega_+)$.

In order to avoid the second derivative term $(\delta p)_{xy}$, we rewrite it into

$$(\delta p)_{xy} = C_1 A + C_2 D_s(\delta p)_x + C_3 D_s(\delta p)_y$$

so that

$$C_i \in C^{1,\alpha}(\Omega_+), \quad i = 1, 2, 3.$$

This can be achieved since

$$(\delta p)_{xy} = \big((c^2-u^2)C_1 + \rho u C_2\big)(\delta p)_{xx} + (-2uvC_1 + \rho v C_2 + \rho u C_3)(\delta p)_{xy} \\ + \big((c^2-v^2)C_1 + \rho v C_3\big)(\delta p)_{yy},$$

that is,

$$\begin{pmatrix} c^2-u^2 & \rho u & 0 \\ -2uv & \rho v & \rho u \\ c^2-v^2 & 0 & \rho v \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since v is close to 0, the determinant of the above coefficient matrix is close to $\rho^2 u^2 c^2 > 0$. Therefore, the linear system is always solvable.

Then (3.10) becomes

$$D_s\big((\delta u)_y + C_2(\delta p)_x + C_3(\delta p)_y\big) - D_s(C_2)(\delta p)_x - D_s(C_3)(\delta p)_y \\ + C_1 A + E_1(\delta u)_y + E_2(\delta p)_x = 0,$$

that is,

$$D_s X + E_3 X + E_4 = 0,$$

where $X = (\delta u)_y + C_2(\delta p)_x + C_3(\delta p)_y$ and $E_3, E_4 \in C^\alpha(\Omega_+)$. Integration along the characteristics of D_s yields

$$\|X\|_{C^\alpha(\Omega_+)} \leqslant C\big(\|X\|_{C^\alpha(\Gamma_1)} + \|E_4\|_{C^\alpha(\Omega_+)}\big).$$

Thus

$$\big\|(\delta u)_y\big\|_{C^\alpha(\Omega_+)} \leqslant C\left(\sum_{i=1}^{3}\|g_i\|_{C^\alpha(\Gamma_1)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\right).$$

Combination of the estimates of $D_s(\delta u)$ and $(\delta u)_y$ yields the same estimate for $(\delta u)_x$. Then

$$\|\delta u\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\Big(\sum \|g_i\|_{C^\alpha(\Gamma_1)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\Big)$$

for some constant $C > 0$. Similarly, we obtain the estimate of $D(\delta v)$ in $C^\alpha(\Omega_+)$ and

$$\|\delta v\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\big(\varepsilon + \varepsilon\|\delta u\|_{C^{1,\alpha}(\Omega_+)} + \|\delta p\|_{C^{1,\alpha}(\Omega_+)}\big).$$

For $\delta\rho$, we find from (3.4) that

$$D_s(\delta\rho) - \frac{1}{c^2}D_s(\delta p) = 0. \tag{3.11}$$

Thus we can obtain the estimate for $\delta\rho$ in $C^{1,\alpha}$ by the similar argument as above.

We note that the constant C in each estimate is independent of the interior points in the region. Thus the estimates can be extended up to the boundary by the limits from the interior. □

Proposition 3.3. *Let δU be the solution determined in Lemmas* 3.1 *and* 3.2. *Then*

$$\|\delta U\|_{C^{1,\alpha}(\Omega_+)} < \delta.$$

Moreover, $\widetilde{U} := \delta U + U_+^0$ is a solution of the linearized equations such that $\widetilde{U} \in \Sigma_\delta$.

Proof. Using estimates (3.8) and (3.9), we have

$$\|\delta U\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\left(\varepsilon + \sum_{i=1,2,3} \|g_i\|_{C^{1,\alpha}(\Gamma_1)} + \varepsilon\left\|U - U_+^0\right\|_{C^{1,\alpha}(\Omega_+)} + \left\|U - U_+^0\right\|^2_{C^{1,\alpha}(\Omega_+)}\right).$$

Together with (3.7), we obtain

$$\|\delta U\|_{C^{1,\alpha}(\Omega_+)} \leqslant C\big(\varepsilon + \left\|U - U_+^0\right\|^2_{C^{1,\alpha}(\Omega_+)}\big). \tag{3.12}$$

Since $U \in \Sigma_\delta$,

$$\|\delta U\|_{C^{1,\alpha}(\Omega_+)} < \delta$$

by taking $\varepsilon < \delta^2$ and $\delta < (3C)^{-1}$ for $C > 1$. $\square$

4. Nonlinear problem and iteration scheme

Based on the estimates of the linearized problem, we now adopt the iteration scheme developed in [5] by Chen and Feldman to establish the existence of solutions of the nonlinear problem.

4.1. Iteration scheme

Given $U \in \Sigma_\delta$, we first locate the shock front ψ by solving (2.8) to determine the boundary ψ, solve the linear problem (3.1)–(3.4) to obtain δU, and finally, extend the solution δU to the whole domain Ω in a fixed way (cf. Section 4.2) such that

$$\|\delta U\|_{C^{1,\alpha}(\Omega)} \leqslant C\|\delta U\|_{C^{1,\alpha}(\Omega_+)}. \tag{4.1}$$

Therefore, we construct a mapping T from Σ_δ to itself by

$$T(U) = \delta U + U_+^0. \tag{4.2}$$

Since Σ_δ is a convex and compact subset of $C^{1,\beta}(\Omega)$ for $\beta < \alpha$, it suffices to prove that T is continuous in order to apply the Schauder fixed point theorem.

4.2. Extension

There are various ways to extend the solution δU. We use the following method to make our extension.

We first consider the extension across a flat boundary for $U \in C^{1,\alpha}$ defined on the half-plane $x \geqslant 0$.

For $x < 0$, we define

$$U(x, y) = aU(-x, y) + bU(-2x, y).$$

In order to obtain the $C^{1,\alpha}$ continuity, we need some condition on the boundary $x = 0$:

$$U(0, y) = aU(0, y) + bU(0, y), \qquad U_x(0, y) = -aU_x(0, y) - 2bU_x(0, y).$$

That is,

$$a = 3, \qquad b = -2$$

so that our extension is

$$U(x, y) = 3U(-x, y) - 2U(-2x, y),$$

and the constant $C = 7$ in (4.1).

For the $C^{n,\alpha}$ extension, set

$$U(x, y) = \sum_{i=0}^{n} c_i U\bigl(-(i+1)x, y\bigr),$$

derive the linear equations for $(c_0, \dots, c_n)$, and solve the linear system to obtain the explicit formula for the extension.

Once we know the way of extension across the flat boundary, we can deal with the curved boundary by flattening. That is, we first do a coordinate transformation to flatten the boundary, then use the formula above to extend the function across the flat boundary and, finally, pull back to the original coordinate system.

4.3. Continuity of the iteration map

We now derive the estimates necessary for the continuity of T.

Given two vector-valued functions U and $\overline{U}$ in Σ_δ such that

$$T(U) = \delta U + U_+^0 \quad \text{and} \quad T(\overline{U}) = \delta\overline{U} + U_+^0,$$

we solve Eq. (2.8) to obtain the corresponding shock fronts ψ and $\bar{\psi}$ which define two downstream regions Ω_+ and $\overline{\Omega}_+$, respectively.

We first construct a region transformation $\mathcal{B} : \Omega_+ \to \overline{\Omega}_+$ given by

$$(\bar{x}, \bar{y}) = \mathcal{B}(x, y) := \bigl(x + \eta(x)\bigl(\bar{\psi}(y) - \psi(y)\bigr), y\bigr),$$

where η is a smooth function with

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1/2, \\ 0 & \text{if } |x| \geqslant 1. \end{cases}$$

Set $U^*(x, y) = \overline{U}(\mathcal{B}(x, y))$ and $\delta U^*(x, y) = \delta\overline{U}(\mathcal{B}(x, y))$.

We first investigate the elliptic equation for the pressure and derive the estimates of the difference of the iterated pressures controlled by the difference of the given functions.

Lemma 4.1. *Suppose that the iterated pressure functions δp and $\delta\bar{p}$ are the solutions for the elliptic equation* (3.1) *with the given functions U and $\overline{U}$, respectively. Then there exists a constant $C > 0$, depending only on the boundary data, such that*

$$\left\|\delta p - \delta p^*\right\|_{C^1(\Omega_+)} \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega)}. \tag{4.3}$$

Proof. As in Lemma 3.1, we write the elliptic equations as

$$a_{ij}(\delta p)_{x_i x_j} + b_i(\delta p)_{x_i} = f, \tag{4.4}$$

$$\bar{a}_{ij}(\delta\bar{p})_{\bar{x}_i\bar{x}_j} + \bar{b}_i(\delta\bar{p})_{\bar{x}_i} = \bar{f}. \tag{4.5}$$

After the coordinate transformation, the elliptic equation (4.5) becomes

$$\hat{a}_{ij}\left(\delta p^*\right)_{x_i x_j} + \hat{b}_i\left(\delta p^*\right)_{x_i} = \hat{f}, \tag{4.6}$$

where

$$\hat{a}_{ij} = \sum_{k,l=1}^{2} a^*_{kl}\frac{\partial x_i}{\partial\bar{x}_k}\frac{\partial x_j}{\partial\bar{x}_l}, \qquad \hat{b}_i = \sum_{k=1}^{2} b^*_k\frac{\partial x_i}{\partial\bar{x}_k} + \sum_{k,j=1}^{2} a^*_{kj}\left(\frac{\partial x_i}{\partial\bar{x}_k}\right)_{\bar{x}_j},$$

$$\hat{f} = -\sum_{i,j,k,l=1}^{2}\frac{\gamma p^*}{\rho^*}\left(\rho^* u^*_i\right)_{x_k}\frac{\partial x_k}{\partial\bar{x}_j}\left(u^*_j\right)_{x_l}\frac{\partial x_l}{\partial\bar{x}_i}$$

for $(u_1, u_2) = (u, v)$.

By definition of the coordinate transformation $\mathcal{B}$, we have

$$\left\|\frac{\partial(\bar{x}, \bar{y})}{\partial(x, y)} - I\right\|_{C^{1,\alpha}(\Omega)} \leqslant C\|U - \overline{U}\|_{C^{1,\alpha}(\Omega)}, \tag{4.7}$$

where I is the identity matrix. Therefore, we have the following estimates:

$$\left\|\hat{a}_{ij} - a^*_{ij}\right\|_{C^{1,\alpha}(\Omega)} \leqslant C\|U - \overline{U}\|_{C^{1,\alpha}(\Omega)}, \tag{4.8}$$

$$\left\|\hat{b}_i - b^*_i\right\|_{L^\infty(\Omega)} \leqslant C\|U - \overline{U}\|_{C^{1,\alpha}(\Omega)}, \tag{4.9}$$

$$\left\|\hat{f} - f^*\right\|_{L^\infty(\Omega)} \leqslant C\|U - \overline{U}\|_{C^{1,\alpha}(\Omega)}. \tag{4.10}$$

Now we take the difference of (4.4) and (4.6) to obtain

$$a_{ij}(\delta p-\delta p^*)_{x_ix_j}+b_i(\delta p-\delta p^*)_{x_i}=F, \tag{4.11}$$

where

$$F=f-\hat f+(\hat a_{ij}-a_{ij})(\delta p^*)_{x_ix_j}+(\hat b_i-b_i)(\delta p^*)_{x_i}.$$

In order to apply the elliptic estimates for the equation of divergence form, we need to rewrite F in the form of $d_0+\sum_j (d_j)_{x_j}$, where

$$\begin{aligned}
&d_0=f-\hat f+(\hat b_i-b_i)(\delta p^*)_{x_i}-(\hat a_{ij}-a_{ij})_{x_j}(\delta p^*)_{x_i},\\
&d_j=(\hat a_{ij}-a_{ij})(\delta p^*)_{x_i}.
\end{aligned}$$

By the $C^{1,\alpha}$ continuity of U and (4.8)–(4.10), we have

$$\begin{aligned}
\|f-\hat f\|_{L^\infty(\Omega)}&\leqslant \|f-\bar f\|_{L^\infty(\Omega)}+\|\bar f-f^*\|_{L^\infty(\Omega)}+\|f^*-\hat f\|_{L^\infty(\Omega)}\\
&\leqslant C\|U-\overline U\|_{C^{1,\alpha}(\Omega)}+\|\bar f\|_{C^\alpha(\Omega)}\|\bar\psi-\psi\|^\alpha_{C^0}\\
&\leqslant C\big(\|U-\overline U\|_{C^{1,\alpha}(\Omega)}+\|U-\overline U\|^\alpha_{C^{1,\alpha}(\Omega)}\big)\\
&\leqslant C\|U-\overline U\|^\alpha_{C^{1,\alpha}(\Omega)}.
\end{aligned}$$

In the same way, we can control the other terms in d_0 and d_j to obtain

$$\|d_0\|_{L^\infty(\Omega)}+\sum_j\|d_j\|_{C^\alpha(\Omega)}\leqslant C\|U-\overline U\|^\alpha_{C^{1,\alpha}(\Omega)}. \tag{4.12}$$

If we use a similar argument to the one above, we also have the estimate for the difference of the boundary data:

$$\|g_2-g_2^*\|_{C^1(\Gamma_1)}\leqslant C\|U-\overline U\|^\alpha_{C^{1,\alpha}(\Omega)},$$

for $g_2^*=\bar g_2\circ\mathcal B$.

Then, similar to Lemma 3.1, we have

$$\begin{aligned}
\|\delta p-\delta p^*\|_{C^1(\Omega_+)}&\leqslant C\bigg(\|g_2-g_2^*\|_{C^1(\Gamma_1)}+\varepsilon\|U-U^*\|_{C^1(\Omega)}+\|d_0\|_{L^\infty(\Omega)}+\sum_j\|d_j\|_{C^\alpha(\Omega)}\bigg)\\
&\leqslant C\|U-\overline U\|^\alpha_{C^{1,\alpha}(\Omega)}.
\end{aligned}$$

This completes the proof. □

The same analysis applies to the rest of the transport equations.

Lemma 4.2. *Let δU and $\delta \overline{U}$ be the extension functions from the solutions of the linear system* (3.1)–(3.4). *Then*

$$\|\delta U - \delta \overline{U}\|_{C^0(\Omega)} \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega)} \tag{4.13}$$

for some constant $C > 0$.

Proof. We use the similar argument as in Lemma 4.1 for the rest of the transport equations. We now focus on Eq. (2.4) to illustrate the method.

Assume that $\delta\bar{u}$ satisfies

$$\bar{\rho}\bar{u}(\delta\bar{u})_{\bar{x}} + \bar{\rho}\bar{v}(\delta\bar{u})_{\bar{y}} + (\delta\bar{p})_{\bar{x}} = 0.$$

We then pull back the variables in the above equation into the region Ω_+ by $\mathcal{B}$ to obtain

$$\left(\rho^* u^* \frac{\partial x}{\partial \bar{x}} + \rho^* v^* \frac{\partial x}{\partial \bar{y}}\right)\big(\delta u^*\big)_x + \rho^* v^*\big(\delta u^*\big)_y + \frac{\partial x}{\partial \bar{x}}\big(\delta p^*\big)_x = 0. \tag{4.14}$$

The difference between (2.4) and (4.14) gives

$$\rho u\big(\delta u - \delta u^*\big)_x + \rho v\big(\delta u - \delta u^*\big)_y + r_1 = 0,$$

where

$$r_1 = (\delta p)_x - \frac{\partial x}{\partial \bar{x}}\big(\delta p^*\big)_x + \left(\rho u - \rho^* u^* \frac{\partial x}{\partial \bar{x}} - \rho^* v^* \frac{\partial x}{\partial \bar{y}}\right)\big(\delta u^*\big)_x + \big(\rho v - \rho^* v^*\big)\big(\delta u^*\big)_y.$$

We know from Lemma 4.1 that

$$\|r_1\|_{C^0(\Omega_+)} \leqslant C\big(\big\|U - U^*\big\|_{C^0(\Omega_+)} + \big\|\delta p - \delta p^*\big\|_{C^1(\Omega_+)}\big) \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega_+)}.$$

Thus, we have

$$\big\|\delta u - \delta u^*\big\|_{C^0(\Omega_+)} \leqslant C\big(\big\|g_1 - g_1^*\big\|_{C^0(\Gamma_1)} + \|r_1\|_{C^0(\Omega_+)}\big) \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega_+)}.$$

Using the same method to the other transport equations, we obtain

$$\big\|\delta U - \delta U^*\big\|_{C^0(\Omega_+)} \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega)}.$$

Thus,

$$\|\delta U - \delta \overline{U}\|_{C^0(\Omega)} \leqslant C\big(\big\|\delta U - \delta U^*\big\|_{C^0(\Omega)} + \big\|\delta U^* - \delta \overline{U}\big\|_{C^0(\Omega)}\big) \leqslant C\|U - \overline{U}\|^{\alpha}_{C^{1,\alpha}(\Omega)}. \qquad \square$$

Remark 4.3. All the estimates for (4.13) hold if we replace the parameter α by β, which is between 0 and α. That is, for any $\beta \in (0, \alpha)$,

$$\|\delta U - \delta \overline{U}\|_{C^0(\Omega)} \leqslant C\|U - \overline{U}\|^{\beta}_{C^{1,\beta}(\Omega)}. \tag{4.15}$$

Based on estimate (4.13), we can prove the continuity of T.

Lemma 4.4. *The mapping T defined by* (4.2) *is continuous in $C^{1,\beta}(\Omega)$.*

Proof. Let

$$T(U^n) = \delta U^n + U_+^0 \quad \text{and} \quad T(U) = \delta U + U_+^0.$$

We need to prove that, for any sequence $\{U^n\}$ converging to U in $C^{1,\beta}$, $\{\delta U^n\}$ with $\|\delta U^n\|_{C^{1,\alpha}(\Omega)} < \delta$ converges to δU in $C^{1,\beta}$.

Suppose that $\{\delta U^n\}$ does not converge to δU in $C^{1,\beta}$. Since $\{\delta U^n + U_+^0\}$ with $\{\delta U^n + U_+^0\} \subset C^{1,\beta}$ is compact for $0 < \beta < \alpha < 1$, there exists a subsequence $\{\delta U^{n_k} + U_+^0\}$ converging to $\delta \widetilde{U} + U_+^0$, which is different from $\delta U + U_+^0$ in $C^{1,\beta}$.

On the other hand, by (4.15), we have

$$\left\|\delta U - \delta U^{n_k}\right\|_{C^0(\Omega)} \leqslant C\left\|U - U^{n_k}\right\|_{C^{1,\beta}(\Omega)}^{\beta}.$$

Letting $k \to \infty$, we conclude

$$\|\delta U - \delta \widetilde{U}\|_{C^0(\Omega)} \leqslant 0,$$

which contradicts the fact that $\delta \widetilde{U}$ and δU are different. □

With the estimates in Sections 4.2 and 4.3, we can prove Theorem 2.2.

4.4. Proof of Theorem 2.2

Since T is continuous, there exists a fixed point δU for T by the Schauder fixed point theorem. Then $U := U_+^0 + \delta U$ is the solution for Problem 2.2.

To prove (2.28), we use estimate (3.12) on the whole domain Ω:

$$\|\delta U\|_{C^{1,\alpha}(\Omega)} \leqslant C\left(\varepsilon + \|\delta U\|_{C^{1,\alpha}(\Omega)}^2\right). \tag{4.16}$$

Since $\|\delta U\|_{C^{1,\alpha}(\Omega)} < \delta$ for $\delta < (2C)^{-1}$, we have

$$\|\delta U\|_{C^{1,\alpha}(\Omega)} \leqslant \frac{C\varepsilon}{1 - C\delta} < 2C\varepsilon.$$

Noticing (2.8), we have

$$\|\psi\|_{C^{2,\alpha}([0,b])} \leqslant C\|v\|_{C^{1,\alpha}(\Omega)}.$$

Therefore, combining this with (2.28), we obtain (2.29).

Remark 4.5. We point out that the fixed point U for the quasi-decoupled system (3.1)–(3.4) is indeed a solution for the original system (2.3)–(2.6). It is obvious that Eqs. (3.2)–(3.4) become

(2.3)–(2.5) for a fixed point U. To recover Eq. (2.6), we conclude from the method we used to generate Eq. (3.2) that

$$(\rho u\partial_x+\rho v\partial_y)(\gamma p u_x+u p_x+\gamma p v_y+v p_y)=0,$$

which implies that the left-hand side of (2.6) equals to a constant along any given fluid trajectory. Since we also impose (2.6) on the right boundary Γ_4, Eq. (2.6) holds along each fluid trajectory, hence on the region Ω_+.

5. Uniqueness of solutions and coordinate transformation

In order to establish the uniqueness of solutions of Problem 2.2, we need to change the coordinate system so that the fluid trajectories are flattened. The coordinate transformation is Euler–Lagrange type (cf. Chen [4]).

We define the new coordinates (ξ,η) by

$$\xi=x,\qquad \eta=\int\limits_{\zeta_2(x)}^{y}(\rho u)(x,s)\,ds, \tag{5.1}$$

so that

$$\eta_y=\rho u,\qquad \eta_x=-\rho v,$$

where the lower limit ζ_2 is the function of the lower boundary defined in (2.17). Hence, we have

$$\partial_x=\partial_\xi-\rho v\partial_\eta,\qquad \partial_y=\rho u\partial_\eta. \tag{5.2}$$

By Eq. (2.3), which comes from the conservation of mass, we conclude that

$$\int\limits_{\zeta_2(x)}^{\zeta_3(x)}\rho u(x,s)\,ds=L\quad \text{is a constant.}$$

Thus, the original domain Ω becomes the following rectangle under the new coordinate system (ξ,η):

$$R=(-1,N)\times(0,L). \tag{5.3}$$

We then transform system (2.3)–(2.6) into

$$u_\xi-\rho v u_\eta+\rho u v_\eta+\frac{u}{\rho}\rho_\xi=0, \tag{5.4}$$

$$\rho u u_\xi+p_\xi-\rho v p_\eta=0, \tag{5.5}$$

$$v_\xi+p_\eta=0, \tag{5.6}$$

$$u_\xi-\rho v u_\eta+\rho u v_\eta+\frac{u}{\gamma p}p_\xi=0. \tag{5.7}$$

Subtraction of (5.4) from (5.7) yields

$$(\ln p - \gamma \ln \rho)_\xi = 0, \tag{5.8}$$

that is,

$$p = A(\eta)\rho^\gamma.$$

We now use (5.5)–(5.8) as our new system in the (ξ, η)-coordinates. We still keep the same notation for the solution variables, but they should be understood in the (ξ, η)-coordinates.

In order to derive the Rankine–Hugoniot relations, we need the equations in conservation form. We rewrite (5.4) and (5.5) into

$$\left(-\frac{1}{\rho u}\right)_\xi + \left(\frac{v}{u}\right)_\eta = 0, \tag{5.9}$$

$$\left(u + \frac{p}{\rho u}\right)_\xi + \left(-\frac{pv}{u}\right)_\eta = 0. \tag{5.10}$$

Then Eqs. (5.6) and (5.8)–(5.10) lead to the Rankine–Hugoniot relations:

$$-\left[\frac{1}{\rho u}\right] = \psi'\left[\frac{v}{u}\right], \qquad \left[u + \frac{p}{\rho u}\right] = -\psi'\left[\frac{pv}{u}\right], \qquad [v] = \psi'[p],$$
$$\left[\frac{u^2+v^2}{2} \frac{\gamma p}{(\gamma-1)\rho}\right] = 0.$$

That is,

$$\psi' = \frac{[v]}{[p]}, \tag{5.11}$$

$$\left[u + \frac{p}{\rho u}\right] = \frac{[v]}{[p]}\left[-\frac{pv}{u}\right], \tag{5.12}$$

$$\left[-\frac{1}{\rho u}\right] = \frac{[v]}{[p]}\left[\frac{v}{u}\right], \tag{5.13}$$

$$\left[\frac{u^2+v^2}{2} \frac{\gamma p}{(\gamma-1)\rho}\right] = 0. \tag{5.14}$$

Our method for uniqueness is similar to the linearization process in Section 3. Following the same way as in Section 3, we differentiate (5.12)–(5.14) along the shock to identify the boundary condition for $W = (u, p, \rho)$, where the corresponding coefficient matrix is

$$B = \begin{pmatrix} 1 - \frac{p}{\rho u^2} & \frac{1}{\rho u} & -\frac{p}{\rho^2 u} \\ \frac{1}{\rho u^2} & 0 & \frac{1}{\rho^2 u} \\ u & \frac{\gamma}{(\gamma-1)\rho} & -\frac{\gamma p}{(\gamma-1)\rho^2} \end{pmatrix}.$$

Since

$$\det B = \frac{c^2 - u^2}{(\gamma - 1)\rho^3 u^3} > 0$$

in the subsonic region, we have the similar condition on the shock

$$(\delta u, \delta p, \delta \rho) = (g_1, g_2, g_3).$$

The elliptic equation for δp is

$$\left(1 - \frac{u^2}{c^2}\right)(\delta p)_{\xi\xi} - 2\rho v(\delta p)_{\xi\eta} + \rho^2\big(u^2 + v^2\big)(\delta p)_{\eta\eta} + f = 0, \tag{5.15}$$

obtained by taking $(\partial_\xi - \rho v \partial_\eta)(5.5) + \rho^2 u^2 \partial_\eta(5.6) - \rho u \partial_\xi(5.7)$, where

$$\begin{aligned} f = {} & (\rho u)_\xi u_\xi - (\rho v)_\xi p_\eta - \rho v u_\xi (\rho u)_\eta + \rho v(\rho v)_\eta p_\eta \\ & + \rho u(\rho v)_\xi u_\eta - \rho u(\rho u)_\xi v_\eta - \rho u\left(\frac{u}{\gamma p}\right)_\xi p_\xi. \end{aligned}$$

The boundary conditions for δp on the lower and upper boundaries are

$$\big(\rho u + \rho v \zeta_j'\big)(\delta p)_\eta = \zeta_j' p_\xi - \rho u^2 \zeta_j'', \quad j = 2, 3. \tag{5.16}$$

The condition on the right boundary is

$$\frac{c^2 - u^2}{c^2}(\delta p)_\xi = \rho v p_\eta + \rho^2 u^2 v_\eta - \rho^2 u v u_\eta. \tag{5.17}$$

Assume now that we have two solutions U_A and U_B in the (ξ, η)-coordinates with the corresponding shocks ψ_A and ψ_B. Let R_A and R_B be the regions bounded by the upper, lower, right boundaries of R, and the shocks Γ_A and Γ_B, respectively. We construct a mapping π from R_A to R_B by

$$\pi : (\xi, \eta) \to (\bar{\xi}, \bar{\eta}) = \left(\frac{N - \psi_B}{N - \psi_A}\xi + N\frac{\psi_B - \psi_A}{N - \psi_A}, \eta\right).$$

Set $U_B^* = U_B \circ \pi$ and $U_D = U_A - U_B^*$. Thus

$$\|\pi - I\|_{C^{1,\alpha}(R_A)} \leqslant C\|\psi_A - \psi_B\|_{C^{1,\alpha}(0,L)} \leqslant C\|U_D\|_{C^{1,\alpha}(R_A)}.$$

Following the same method for the existence part in Section 4, we obtain the elliptic equation for p_D

$$\big(a_{ij}(p_D)_{\xi_i}\big)_{\xi_j} + b_j(p_D)_{\xi_j} = d_0 + (d_i)_{\xi_i}, \tag{5.18}$$

where $(\xi_1, \xi_2) = (\xi, \eta)$.

For the uniqueness, we assume (2.25), which gives us better control on the boundary for U_D.

For the boundary condition for p_D on Γ_A, we know that

$$p_D|_{\Gamma_A} = G_2(U_-, U_A) - G_2\big(U_- \circ \pi, U_B^*\big) \equiv g_D,$$

where G_2 is a functional defined when the boundary conditions on the shock are derived.

Notice that

$$U_- \circ \pi(\xi,\eta) - U_-(\xi,\eta) = (\bar\xi - \xi)\int_0^1 (U_-)_\xi\big(t(\bar\xi - \xi) + \xi, \eta\big)\,dt.$$

Therefore, we have

$$\begin{aligned}
\|g_D\|_{C^{1,\alpha}(0,L)} &\leqslant \big\|G_2(U_-, U_A) - G_2\big(U_-, U_B^*\big)\big\|_{C^{1,\alpha}(R_A)} \\
&\quad + \big\|G_2\big(U_- \circ \pi, U_B^*\big) - G_2\big(U_-, U_B^*\big)\big\|_{C^{1,\alpha}(R_A)} \\
&\leqslant C\delta\|U_D\|_{C^{1,\alpha}(R_A)} + C\|U_- \circ \pi - U_-\|_{C^{1,\alpha}(R_A)} \\
&\leqslant C\delta\|U_D\|_{C^{1,\alpha}(R_A)} + C\big\|(U_-)_\xi\big\|_{C^{1,\alpha}(R_A)}\|\psi_B - \psi_A\|_{C^{1,\alpha}(0,L)} \\
&\leqslant C(\varepsilon + \delta)\|U_D\|_{C^{1,\alpha}(R_A)}.
\end{aligned}$$

We treat the other boundary conditions in a similar way. Therefore, we obtain

$$\|p_D\|_{C^{1,\alpha}(R_A)} \leqslant C\delta\|U_D\|_{C^{1,\alpha}(R_A)}. \tag{5.19}$$

We apply the same technique as in Lemma 3.2 with Eq. (5.15) to obtain the estimates for the rest of the solution variables. Notice that there is a difference between the new and old coordinate systems. Using the (ξ,η)-coordinates, we get better control on the transport equations. The fluid trajectories for the two solutions actually merge into the same straight lines $\eta = C$ in the new coordinate system.

The analysis of the transport equations is similar. Without loss of generality, we investigate only Eq. (5.5) here.

Equation (5.5) for u_A can be written as

$$(u_A)_\xi + \frac{1}{\rho_A u_A}(p_A)_\xi - \frac{v_A}{u_A}(p_A)_\eta = 0. \tag{5.20}$$

The equation for u_B^* is

$$\big(u_B^*\big)_\xi + \left(\frac{1}{\rho_B^* u_B^*} + \frac{\partial\bar\xi}{\partial\eta}\frac{v_B^*}{u_B^*}\right)\big(p_B^*\big)_\xi - \frac{\partial\bar\xi}{\partial\xi}\frac{v_B^*}{u_B^*}\big(p_B^*\big)_\eta = 0. \tag{5.21}$$

The difference between (5.21) and (5.20) gives

$$(u_D)_\xi + \frac{1}{\rho_A u_A}(p_D)_\xi - \frac{v_A}{u_A}(p_D)_\eta + a_1\big(p_B^*\big)_\xi + a_2\big(p_B^*\big)_\eta = 0, \tag{5.22}$$

where

$$a_1 = \frac{1}{\rho_B^* u_B^*} - \frac{1}{\rho_A u_A} + \frac{\partial \bar{\xi}}{\partial \eta} \frac{v_B^*}{u_B^*}, \qquad a_2 = \frac{v_A}{u_A} - \frac{v_B^*}{u_B^*} \frac{\partial \bar{\xi}}{\partial \xi}.$$

From (5.22), it is easy to see that

$$\big\| \big(u_D, (u_D)_\xi\big) \big\|_{C^\alpha(R_A)} \leqslant C\big(\|p_D\|_{C^{1,\alpha}(R_A)} + \big\|(a_1, a_2)\big\|_{C^\alpha(R_A)} \big\|p_B^*\big\|_{C^{1,\alpha}(R_A)} \big).$$

Since $\|(a_1, a_2)\|_{C^\alpha(R_A)} \leqslant C\|U_D\|_{C^\alpha(R_A)}$, together with (5.19), we obtain

$$\big\| \big(u_D, (u_D)_\xi\big) \big\|_{C^\alpha(R_A)} \leqslant C\delta \|U_D\|_{C^{1,\alpha}(R_A)}. \tag{5.23}$$

The remainder is to estimate $(u_D)_\eta$. We follow the same way as in Lemma 3.2.

First, differentiating (5.22) with respect to η, we have

$$\big((u_D)_\eta\big)_\xi + b_1 (p_D)_{\xi\eta} + b_2 (p_D)_{\eta\eta} + a_1 \big(p_B^*\big)_{\xi\eta} + a_2 \big(p_B^*\big)_{\eta\eta} + b_3 = 0$$

with

$$\big\|(b_1, b_2)\big\|_{C^{1,\alpha}(R_A)} \leqslant C, \qquad \|b_3\|_{C^\alpha(R_A)} \leqslant C\delta \|U_D\|_{C^{1,\alpha}(R_A)}.$$

Furthermore, by the structure of (5.15),

$$(p_D)_{\eta\eta} = c_1 (p_D)_{\xi\xi} + c_2 (p_D)_{\xi\eta} + c_3, \qquad \big(p_B^*\big)_{\eta\eta} = d_1 \big(p_B^*\big)_{\xi\xi} + d_2 \big(p_B^*\big)_{\xi\eta} + d_3$$

with

$$\big\|(c_1, c_2, d_1, d_2)\big\|_{C^{1,\alpha}(R_A)} \leqslant C, \qquad \big\|(c_3, d_3)\big\|_{C^\alpha(R_A)} \leqslant C\delta \|U_D\|_{C^{1,\alpha}(R_A)}.$$

Let

$$X = (u_D)_\eta + (b_1 + b_2 c_2)(p_D)_\eta + (a_1 + a_2 d_2)\big(p_B^*\big)_\eta + b_2 c_1 (p_D)_\xi + a_2 d_1 \big(p_B^*\big)_\xi.$$

Thus we obtain the following equation

$$X_\xi + E = 0$$

with

$$\|E\|_{C^\alpha(R_A)} \leqslant C\delta \|U_D\|_{C^{1,\alpha}(R_A)}.$$

We integrate the above equation with respect to ξ, together with the boundary condition for u_D, to have

$$\|u_D\|_{C^{1,\alpha}(R_A)} \leqslant C\delta \|U_D\|_{C^{1,\alpha}(R_A)}. \tag{5.24}$$

The only difference between the equations for u_D and v_D is the boundary conditions. For v_D, we impose the boundary condition on $x = N$. In the (ξ, η)-coordinates, we have

$$v|_{\xi=N} = h\left(\int_0^\eta \frac{1}{\rho u}(N,s)\,ds\right).$$

Hence, we obtain

$$\begin{aligned} v_D(N,\eta) = {} & \int_0^\eta \left(\frac{1}{\rho_A u_A} - \frac{1}{\rho_B u_B}\right)(N,s)\,ds \\ & \times \int_0^1 h'\left(\int_0^\eta \left(t\left(\frac{1}{\rho_A u_A} - \frac{1}{\rho_B u_B}\right) + \frac{1}{\rho_B u_B}\right)(N,s)\,ds\right)dt. \end{aligned}$$

Therefore,

$$\left\| v_D(N,\cdot)\right\|_{C^{1,\alpha}(0,L)} \leqslant C\|h\|_{C^{2,\alpha}(0,L)}\|U_D\|_{C^{1,\alpha}(R_A)} \leqslant C\varepsilon\|U_D\|_{C^{1,\alpha}(R_A)}.$$

Finally, we obtain the inequality

$$\|U_D\|_{C^{1,\alpha}(R_A)} \leqslant C\delta\|U_D\|_{C^{1,\alpha}(R_A)}.$$

For sufficiently small $\delta > 0$, we conclude

$$\|U_D\|_{C^{1,\alpha}(R_A)} \leqslant \frac{1}{2}\|U_D\|_{C^{1,\alpha}(R_A)},$$

which implies $U_D = 0$. This establishes the uniqueness in Theorem 2.2.

6. Solutions in the supersonic region

In this section we establish Theorem 2.1 for the existence and uniqueness of supersonic solutions in the upstream region that is the rectangle $R = (-1,1)\times(0,b)$ as required in Sections 2–5. We can obtain the same results for the region with a curved boundary which requires only a little more notational complexity.

First, system (2.1) can be rewritten into the nondivergence form:

$$AU_x + BU_y = 0, \tag{6.1}$$

where

$$A = \begin{pmatrix} \rho & 0 & 0 & u \\ \rho u & 0 & 1 & 0 \\ 0 & \rho u & 0 & 0 \\ 1 & 0 & \frac{u}{\gamma p} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \rho & 0 & v \\ \rho v & 0 & 0 & 0 \\ 0 & \rho v & 1 & 0 \\ 0 & 1 & \frac{v}{\gamma p} & 0 \end{pmatrix}. \tag{6.2}$$

By solving $\det(B - \lambda A) = 0$, we find the eigenvalues of (2.1):

$$\lambda_1 = \lambda_2 = \frac{v}{u}, \qquad \lambda_{3,4} = \frac{uv \pm c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}.$$

The corresponding left eigenvectors are

$$l_1 = (1, 0, 0, -\rho), \qquad l_2 = (0, u, v, 0), \qquad l_{3,4} = \big(0, \lambda_{3,4}, -1, \rho(v - \lambda_{3,4}u)\big).$$

The initial-boundary conditions are (2.19) and (2.20) satisfying (2.22).

We use the characteristic method under the $C^{1,\alpha}$ framework. The similar results and estimates for the Cauchy problem can be found in [12]. Since we deal with the initial–boundary value problem with the slip boundary condition here, we will mainly investigate the part different from the Cauchy problem as in [12].

Now we know $v = 0$ on the boundary $y = 0, b$. We need three more conditions on the boundary to prescribe u, p, and ρ. We use the characteristic method to determine the data on the boundary. We only analyze the data on the lower boundary $y = 0$.

Set $R_s = (-1, s) \times (0, b)$. We study the local solutions on R_δ first for small δ.

Define the ith characteristics f_i passing through $(x, 0)$ by the ordinary differential equation

$$\frac{df_i(\tau; x, 0)}{d\tau} = \lambda_i\big(U(\tau, f_i)\big), \tag{6.3}$$

$$f_i(x; x, 0) = 0. \tag{6.4}$$

Multiplication of (6.1) by l_i yields

$$l_i A(U_x + \lambda_i U_y) = 0.$$

Thus, along the ith characteristics, we have

$$\frac{d}{d\tau}(l_i AU) = \frac{d}{d\tau}(l_i A)U. \tag{6.5}$$

We know $\lambda_{1,2} = 0$ and $\lambda_4 < 0$ on $y = 0$. Therefore, the 1st, 2nd, and 4th characteristics can travel to the left and reach the initial boundary $x = -1$. Let $\xi_i(x) = f_i(0; x, 0)$. Integrate (6.5) along the characteristics to obtain

$$l_i AU(x, 0) = l_i A\Phi_0\big(\xi_i(x)\big) + \int_{-1}^{x} \frac{d}{d\tau}(l_i A)U \, d\tau \equiv \chi_i. \tag{6.6}$$

Now we linearize these conditions as follows: in (6.6), we keep the right-hand side. On the left-hand side, we replace the given solution variables U by $\overline{U}$, which are unknown variables, i.e.,

$$l_i A\overline{U}(x, 0) = \chi_i. \tag{6.7}$$

Together with the condition $v(x, 0) = \bar{v}(x, 0) = 0$, we have

$$-\frac{u}{c^2}\bar{p} + u\bar{\rho} = \chi_1, \qquad \rho u^2\bar{u} + u\bar{p} = \chi_2, \qquad -\lambda_4\left(1 - \frac{u^2}{c^2}\right)\bar{p} = \chi_4.$$

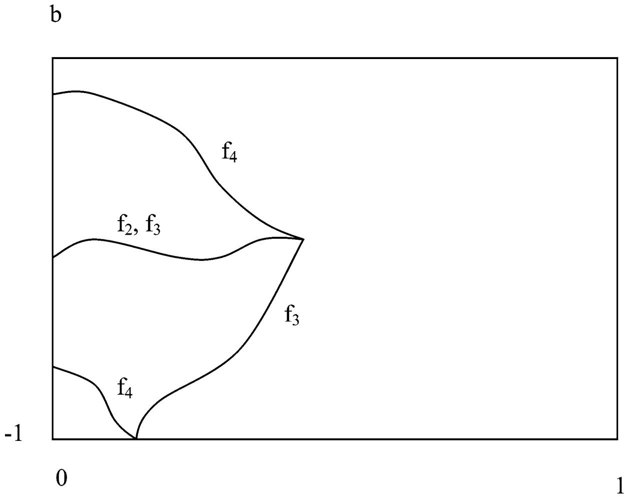

Fig. 2. Characteristics in the domain.

It is obvious that $\overline{U}$ can be written as a linear combination of χ_i with the functions of U as its coefficients. Therefore, for any $(x, y) \in R_\delta$, $\overline{U}(x, y)$ can be determined along the characteristics tracing back either to the upper or lower boundary, or to the initial boundary. Then all the estimates in [12] apply to our problem here. By [12, Theorems 4.1, 4.2], we conclude the local existence and uniqueness of solutions to our hyperbolic system as follows.

Lemma 6.1. *For the initial–boundary value problem* (6.1), (2.19), *and* (2.20) *with*

$$\|\Phi_0 - U_-^0\|_{C^{1,\alpha}(0,b)} < \hat{\varepsilon}$$

for $\hat{\varepsilon} \in (0, 1)$, *there exists small* $\delta_* > 0$ *depending only on* $\hat{\varepsilon}$, U_-^0, *and* δ_0 *such that there exists a unique solution* U *on* R_{δ_*} *with the following estimate*:

$$\|U - U_-^0\|_{C^{1,\alpha}(R_{\delta_*})} \leqslant C_1 \|\Phi_0 - U_-^0\|_{C^{1,\alpha}(0,1)}, \tag{6.8}$$

where C_1 *depends only on* U_-^0, δ_0, *and* δ_*. *In addition, for the initial–boundary value problem* (6.1), (2.19), *and* (2.20) *satisfying*

$$\|\Phi_0 - U_-^0\|_{C^{2,\alpha}(0,b)} < \hat{\varepsilon},$$

we have

$$\|U - U_-^0\|_{C^{2,\beta}(R_{\delta_*})} \leqslant C_2 \|\Phi_0 - U_-^0\|_{C^{2,\beta}(0,1)}$$

for any $\beta \in [0, 1]$.

We will extend the local solution by control of the initial data. The result is stated as the following theorem.

Theorem 6.2. *For the initial-boundary value problem* (6.1), (2.19), *and* (2.20) *satisfying* (2.22), *there exists a constant $\varepsilon_0 > 0$ depending only on δ_* and C_1 such that, when $\varepsilon \leqslant \varepsilon_0$, there exists a unique solution U on Ω satisfying* (2.23) *and* (2.24). *In addition, for the initial-boundary value problem* (6.1), (2.19), *and* (2.20) *satisfying* (2.25),

$$\left\| U - U_-^0 \right\|_{C^{2,\beta}(R_{\delta_*})} \leqslant C_3 \left\| \Phi_0 - U_-^0 \right\|_{C^{2,\beta}(0,1)} \tag{6.9}$$

for any $\beta \in [0, 1]$.

Proof. We fix the constants $\hat{\varepsilon}$ and δ_* in Lemma 6.1 and set $\varepsilon = \hat{\varepsilon}/C_1^{2/\delta_*+1}$.

We apply Lemma 6.1 to obtain R_{δ_*} as the existence region for U and the estimate

$$\left\| U - U_-^0 \right\|_{C^{1,\alpha}(R_{\delta_*})} \leqslant C_1 \left\| \Phi_0 - U_-^0 \right\|_{C^{1,\alpha}(0,1)} < \hat{\varepsilon}/C_1^{2/\delta_*}.$$

We assume $C_1 > 1$ so that

$$\left\| U(\delta_*, \cdot) - U_-^0 \right\|_{C^{1,\alpha}(0,1)} < \hat{\varepsilon}.$$

We then use $U(\delta_*, y)$ as our new initial data to solve system (6.1). By applying Lemma 6.1 up to $[2/\delta_*] + 1$ times, we can extend the local solution to the region R_1, together with estimate (6.8).

In addition, for the initial–boundary value problem (6.1), (2.19), and (2.20) satisfying (2.25), we can show (6.9) in a similar way.

Finally, we investigate the condition in (2.24), i.e.,

$$(u, p, \rho)_y|_{O \cup J} = 0. \tag{6.10}$$

This can be seen as follows. From Eq. (2.5) and the fact that $v|_{y=0,b} = 0$, we know that $p_y|_{y=0,b} = 0$.

On the lower or upper boundary $y = 0$ or $y = b$, which is a fluid trajectory, we have $\rho = A(y)p^{1/\gamma}$. By the assumption

$$(p_0)_y = (\rho_0)_y = 0 \quad \text{at } y = 0, b,$$

we find

$$\rho_0'(0) = A'(0)p(0) + A(0)p'(0) = 0.$$

Therefore, $A'(0) = 0$. Also, $A'(b) = 0$. Thus we have

$$\rho_y|_{y=0,b} = \left(p^{\frac{1}{\gamma}} A' + \frac{1}{\gamma} p^{\frac{1}{\gamma}-1} p_y A \right)\bigg|_{y=0,b} = 0.$$

Now we prove $u_y|_{y=0,b} = 0$. Since we assume that the initial data Φ_0 is C^2, the solution U is also C^2 in the region R_1. Then we can differentiate Eq. (2.4) with respect to y to obtain

$$\rho u(u_y)_x + \rho v(u_y)_y + (\rho u)_y u_x + (\rho v)_y u_y + (p_y)_x = 0.$$

Use of $p_y|_{y=0,b} = 0$ yields $(p_y)_x|_{y=0,b} = 0$. Also, by using Eq. (2.3) and $v|_{y=0,b} = 0$, the above equation is reduced to

$$(u_y)_x = \frac{\rho_x}{\rho} u_y \quad \text{on } y = 0, b.$$

We solve this equation to obtain

$$u_y = (u_0)_y \frac{\rho}{\rho_0} = 0,$$

since $(u_0)_y|_{y=0,b} = 0$. Therefore, we conclude $(u, p, \rho)_y|_{y=0,b} = 0$. $\square$

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References

[1] S. Canic, B.L. Keyfitz, E.H. Kim, A free boundry problems for a quasilinear degenerate elliptic equation: Transonic regular reflection, Comm. Pure Appl. Math. 55 (2002) 71–92.
[2] S. Canic, B.L. Keyfitz, E.H. Kim, Free boundary problems for nonlinear wave systems: Mach stems for interacting shocks, SIAM J. Math. Anal. 37 (2005) 1947–1977.
[3] S. Canic, B.L. Keyfitz, G. Lieberman, A proof of existence of perturbed steady transonic shocks via a free boundary problem, Comm. Pure Appl. Math. 53 (2000) 484–511.
[4] G.-Q. Chen, The method of quasidecoupling for discontinuous solutions to conservation laws, Arch. Ration. Mech. Anal. 121 (1992) 131–185.
[5] G.-Q. Chen, M. Feldman, Mulitdimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, J. Amer. Math. Soc. 3 (2003) 461–494.
[6] G.-Q. Chen, M. Feldman, Steady transonic shocks and free boundary problems for the Euler equations in infinite cylinders, Comm. Pure Appl. Math. 57 (2004) 310–356.
[7] G.-Q. Chen, M. Feldman, Free boundary problems and transonic shocks for the Euler equations in unbounded domains, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) 3 (2004) 827–869.
[8] G.-Q. Chen, M. Feldman, Existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections, Arch. Ration. Mech. Anal. (2006), in press.
[9] S. Chen, Stability of transonic shock front in multi-dimensional Euler system, Trans. Amer. Math. Soc. 357 (2005) 287–308.
[10] S. Chen, Stability of Mach configuration, Comm. Pure Appl. Math. 59 (2006) 1–35.
[11] R. Courant, K.O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948.
[12] T. Li, W. Yu, Boundary value problems for quasilinear hyperbolic systems, Duke Univ. Math. Ser. V (1985).
[13] A.G. Kuzmin, Boundary Value Problems for Transonic Flow, Wiley, New York, 2002.
[14] Y. Zheng, A global solution to a two-dimensional Riemann problem involving shocks as free boundaries, Acta Math. Appl. Sinica (English Ser.) 19 (2003) 559–572.
[15] Y. Zheng, Two-dimensional regular shock reflection for the pressure-gradient system of conservation laws, Acta Math. Appl. Sinica (English Ser.) 22 (2006) 177–210.